

Necessary Conditions for Infinite Horizon Optimal Control Problems Revisited

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Abstract

Necessary optimality conditions in the form of the maximum principle for control problems with infinite time horizon are considered. Both finite and infinite values of objective functional are allowed since the concept of overtaking and weakly overtaking optimality is used. New form of optimality condition is obtained and compared with the transversality conditions usually used in the literature. The examples, where these transversality conditions may fail while the new condition holds are presented. For Ramsey problem of capital accumulation a simple form of necessary optimality conditions is derived, which is also valid in the case of zero discounting.

1 Introduction

Optimal control problems with infinite horizon play an important role in economic theory. For instance, in the theory of economic growth, Pontragin's maximum principle is the workhorse for many researchers. The proof of the maximum principle for infinite time horizon one can find, e.g., in [6]. The proved theorem does not include transversality conditions. Moreover it is known, [6, 7], that usually used forms of transversality conditions:

$$\lim_{t \rightarrow \infty} \psi(t) = 0, \quad (1)$$

$$\lim_{t \rightarrow \infty} \langle \hat{x}(t), \psi(t) \rangle = 0, \quad (2)$$

can be not necessary, where \hat{x} is the optimal state trajectory, ψ is the corresponding adjoint variable, and brackets $\langle \cdot, \cdot \rangle$ denote scalar product of two vectors.

Transversality condition obtained in [8], under assumptions including that the objective functional takes only finite values, has the form of Hamiltonian \mathcal{H} converging to zero

$$\lim_{t \rightarrow \infty} \mathcal{H}(\hat{x}(t), \hat{u}(t), t, \psi(t)) = 0, \quad (3)$$

where \hat{x} and \hat{u} are the optimal state trajectory and control.

In [1, 2, 3] the authors determine the adjoint variable uniquely by a Cauchy-type formula. This formula is equivalent to the transversality conditions in the following form

$$\lim_{t \rightarrow \infty} Y(t) \psi(t) = 0, \quad (4)$$

where $Y(t)$ is the fundamental matrix of the state equation linearized about the optimal solution.

Due to their limitations all aforementioned transversality conditions fail to select the optimal solution of Ramsey problem without discounting, [9], where we consider diverging objective functional. Condition (3) can hold if we modify the objective improper integral subtracting from its integrand the constant, such that for optimal solution the integral converges, see e.g., [5, Section 7]. Condition (4) fails because it is proved for problems without state constraints, while Ramsey requires the state variable (capital) to be not negative for all time instances. This is probably why (1) and (2) also fail in Ramsey problem without discounting.

It is technically difficult in general to take explicitly into account the state constraints. One way out, like for non-negativity of capital in Ramsey model, is to consider trajectories which do not violate state constraints and satisfy optimality conditions. It can be done with the necessary conditions obtained in this paper, because these conditions do not contain explicitly the adjoint variable. Thus, it is possible to select the optimal among feasible trajectories in Ramsey problem.

The paper in hand considers the examples without state constraint too, where conditions (1)–(4) also do not hold, while the new conditions are valid. The proved conditions include condition (4) as a special case and extend its domain of applicability.

2 Statement of the problem

Let X be a nonempty open convex subset of R^n , U be an arbitrary nonempty set in R^m . Let us consider the following optimal control problem:

$$\int_{t_0}^{\infty} g(x(t), u(t), t) dt \rightarrow \max_u \quad (5)$$

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0, \quad (6)$$

where $x(t) \in X$ and $u(t) \in U$. Functions f and g are differentiable w.r.t. their first argument, x .

3 Concept of optimality

Let us introduce the following functional, subject to (6),

$$J(u(\cdot), x_0, t_0, T) = \int_{t_0}^T g(x(t), u(t), t) dt, \quad (7)$$

that may be unbounded when time horizon T approaches infinity, $T \rightarrow \infty$.

Definition 3.1. An admissible control $\hat{u}(\cdot)$ for which the corresponding trajectory $\hat{x}(\cdot)$ exists on $[t_0, +\infty)$ is overtaking optimal (OO) if for every $\varepsilon > 0$ there exists $T = T(\varepsilon, u(\cdot)) > t_0$ such that for all $T' \geq T$ the corresponding admissible trajectory $x(\cdot)$ is either non-extendible to $[0, T']$ in X or

$$J(u(\cdot), x_0, t_0, T') - J(\hat{u}(\cdot), x_0, t_0, T') \leq \varepsilon. \quad (8)$$

Definition 3.2. An admissible control $\hat{u}(\cdot)$ for which the corresponding trajectory $\hat{x}(\cdot)$ exists on $[t_0, +\infty)$ is weakly overtaking optimal (WOO) if for every $\varepsilon > 0$ and $T > t_0$ one can find $T' = T'(\varepsilon, T, u(\cdot)) \geq T$ such that the corresponding admissible trajectory $x(\cdot)$ is either non-extendible to $[0, T']$ in X or

$$J(u(\cdot), x_0, t_0, T') - J(\hat{u}(\cdot), x_0, t_0, T') \leq \varepsilon. \quad (9)$$

4 Optimality conditions

With the use of the adjoint variable ψ we introduce *Hamiltonian*

$$\mathcal{H}(x, u, t, \psi, \lambda) = \lambda g(x, u, t) + \langle \psi, f(x, u, t) \rangle \quad (10)$$

where brackets $\langle \cdot, \cdot \rangle$ denote scalar product of two vectors. It is known, see [6], that there exist scalar $\lambda \geq 0$ and vector ψ_0 , such that $(\lambda, \psi_0) \neq 0$ and the maximum principle holds:

$$\mathcal{H}(\hat{x}(t), \hat{u}(t), t, \psi(t), \lambda) = \max_{u \in U} \mathcal{H}(\hat{x}(t), u, t, \psi(t), \lambda), \quad (11)$$

along with the adjoint equation:

$$-\dot{\psi}(t) = \frac{\partial \mathcal{H}}{\partial x}(\hat{x}(t), \hat{u}(t), t, \psi(t), \lambda), \quad \psi(t_0) = \psi_0. \quad (12)$$

In order to select such λ and ψ_0 with the use of transversality condition we consider the linearization of system (6)

$$\dot{y}(t) = \left(\frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t) \right) y(t). \quad (13)$$

Its solution, for the given initial condition $y(\tau)$, can be written with the use of the *state-transition matrix* $K(t, \tau)$:

$$y(t) = K(t, \tau) y(\tau). \quad (14)$$

The following assumption determines how well linearized system (13) should approximate the original system, so that the maximum principle holds under the proposed transversality condition.

Assumption 1. *For almost all time instances $\tau \geq t_0$ directional derivative $\langle J_x, \zeta \rangle$ of the functional at the optimal trajectory \hat{x} uniformly bounds from below the corresponding limit :*

$$\lim_{\alpha \rightarrow 0} \inf_{T \in [\tau, \infty)} \left(\frac{J(\hat{u}(\cdot), \hat{x}(\tau) + \alpha \zeta, \tau, T) - J(\hat{u}(\cdot), \hat{x}(\tau), \tau, T)}{\alpha} - \langle \hat{J}_x(\tau, T), \zeta \rangle \right) \geq 0,$$

with all perturbations of the initial conditions, $x(\tau) = \hat{x}(\tau) + \alpha \zeta$, such that the resulting trajectories are feasible, $x(t) \in X$ in $[\tau, \infty)$, where we denote

$$\hat{J}_x(\tau, T) := \int_{\tau}^T K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t) dt. \quad (15)$$

Remark 1. If functions f and g are linear w.r.t. x , then Assumption 1 holds as equality for any scalar $\alpha > 0$ and direction ζ .¹

Assumption 1 states that when $\alpha \rightarrow 0$, the perturbation of the functional is uniformly (in T) bounded from below by its linear approximation.

Proposition 4.1 (General transversality conditions). *Let Assumption 1 be fulfilled and the absolute value of $\hat{J}_x(\tau, T)$ bounded in T , i.e. there exists function $M(\tau) > 0$ such that for all $T \geq \tau$ we have*

$$\left| \hat{J}_x(\tau, T) \right| \leq M(\tau). \quad (16)$$

1) *If control \hat{u} is WOO, then for all $\tau \in [t_0, \infty)$ and $u \in U$*

$$\liminf_{T \rightarrow \infty} \left(\mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \right) \leq 0, \quad (17)$$

2) *If control \hat{u} is OO, then for all $\tau \in [t_0, \infty)$ and $u \in U$*

$$\limsup_{T \rightarrow \infty} \left(\mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \right) \leq 0. \quad (18)$$

Proof. See Appendix A. □

Under additional assumption, that J_x converges as $T \rightarrow \infty$, referred to as condition of dominating discount in general form, see e.g. [3], the following corollary proves that maximum principle holds in normal case ($\lambda = 1$) and provides an explicit expression for the adjoint variable, which is equivalent to transversality condition (4).

Corollary 1. *If control \hat{u} is WOO, Assumption 1 holds, and the following limit exists*

$$\hat{\psi}(\tau) := \lim_{T \rightarrow \infty} \hat{J}_x(\tau, T) = \int_{\tau}^{\infty} K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t) dt, \quad (19)$$

see (15), then $\hat{\psi}$ solves adjoint system (12) in the normal case ($\lambda = 1$) and the maximum principle holds:

$$\mathcal{H}(\hat{x}(t), \hat{u}(t), t, \hat{\psi}(t), 1) \leq \mathcal{H}(\hat{x}(t), u, t, \hat{\psi}(t), 1), \quad \text{for all } u \in U. \quad (20)$$

¹Due to inequality in Assumption 1, it is also satisfied when g is convex and f is linear w.r.t. x .

Proof. Conditions (17) and (18) take the form of maximum principle (20). Differentiation w.r.t. τ of the integral expression for vector-function $\hat{\psi}$ in (19) shows that this is a solution of the adjoint system (12) in the normal case. Alternatively, see Proposition 4.2 for $a_0 = 0$ and $\lambda = 1$. \square

The following proposition allows not only for normal case ($\lambda = 1$), but also for abnormal one, when $\lambda = 0$.

Proposition 4.2 (Special transversality conditions). *If control \hat{u} is WOO, Assumption 1 holds, and the following limit exists*

$$\lim_{T \rightarrow \infty} K^*(T, t_0) \psi(T) = a_0, \quad (21)$$

where ψ is the solution of adjoint equation (12) such that maximum condition (11) is fulfilled, then the limit in (19) exists and $\psi(\tau) = K^*(t_0, \tau) a_0 + \lambda \hat{\psi}(\tau)$.²

Proof. See Appendix B. \square

Corollary 2. *The limit in (19) exists if, and only if, (4) holds.*

Proof. Taking $a_0 = 0$ in case 1) of Proposition 4.2, we have

$$\lim_{T \rightarrow \infty} K^*(T, t_0) \hat{\psi}(T) = \lim_{T \rightarrow \infty} Y(T) \hat{\psi}(T) = 0, \quad (22)$$

where the state transition matrix $K(T, \tau) = Y(T) Y(\tau)^{-1}$ is expressed via the non-degenerate fundamental matrix Y , such that $Y(t_0) = I$. \square

Remark 2. In the case of additional state constraints, consider set $G \subset U \times X$ of all feasible pairs (u, x) which belong to the trajectories satisfying maximum principle (11) and adjoint equation (12) and not violating at any time $\tau \in [t_0, \infty)$ any constraints imposed. Then, in Proposition 4.1, instead of $u \in U$ we should take $u \in \{u : (u, x(\tau)) \in G\}$.

When in addition integrand g does not depend on state variable x , like in Ramsey model in Example 5.1, then Assumption 1 holds and $\hat{J}_x(\tau, T) \equiv 0$. Necessary conditions in Proposition 4.1 take the following form

$$g(\hat{x}(t), \hat{u}(t), t) \geq g(\hat{x}(t), u, t), \quad \text{for all } u \in \{u : (u, x(t)) \in G\}. \quad (23)$$

²The similar expression was obtained in [1, Section 6] under certain assumptions ensuring the existence of the limit in (19). It was proved that vector a_0 has nonnegative components when the problem is autonomous and monotonic in the state variable, i.e. $\frac{\partial g}{\partial x}(x, u) > 0$, $\frac{\partial f}{\partial x}(x, u) > 0$ for all $(x, u) \in X \times U$ and for all optimal trajectories \hat{x} there exist $\tau \geq t_0$ and vector u_τ such that $f(\hat{x}(t), u_\tau) > 0$.

5 Examples

Example 5.1 (centralized Ramsey model without discounting). We maximize aggregated constant relative risk aversion utility

$$\int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} dt \rightarrow \max_{c>0}, \quad \text{s.t.} \quad \dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t), \quad k(t) \geq 0,$$

where $k(0) = k_0 > 0$, $\theta \neq 1$, $\theta > 0$, and $\alpha \in (0, 1)$. Hamiltonian reads as

$$\mathcal{H}(k, c, t, \psi, 1) = \frac{c^{1-\theta}}{1-\theta} + \psi (k^\alpha - \delta k - c)$$

Stationarity condition $c(t)^{-\theta} = \psi(t)$ and the adjoint equation $-\dot{\psi}(t) = (\alpha k(t)^{\alpha-1} - \delta) \psi(t)$ result in the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha k(t)^{\alpha-1} - \delta}{\theta}.$$

It follows from the Euler equation and the state equation, that any feasible pair (k, c) , that does not violate constraint $k(t) \geq 0$, converges either to steady state (k_*, c_*) , where $k_* = (\delta/\alpha)^{\frac{1}{\alpha-1}}$ and $c_* = (1-\alpha)(\delta/\alpha)^{\frac{\alpha}{\alpha-1}} > 0$, or to $(\delta^{\frac{1}{\alpha-1}}, 0)$, where $k_* < \delta^{\frac{1}{\alpha-1}}$. Necessary conditions (23) formulated in Remark 2 single out the optimal pair converging to (k_*, c_*) .

$$\frac{\hat{c}(t)^{1-\theta}}{1-\theta} \geq \frac{c^{1-\theta}}{1-\theta}, \quad \text{for all } c \in \{c : (c, k(t)) \in G\}, \quad (24)$$

where G is the set of the trajectories governed by the state and Euler equations, not violating conditions $k(t) \geq 0$, see solid lines in Figure 1.

The following example of an autonomous monotonic problem illustrates Proposition 4.2.

Example 5.2.

$$\int_0^\infty e^{-\rho t} x(t) dt \rightarrow \max_u, \quad \text{s.t.} \quad \dot{x}(t) = u(t), \quad x(0) = 0, \quad u(t) \in [0, 1]$$

where obvious optimal control is $u \equiv 1$.

For $\rho > 0$ we have from Proposition 4.2 case 1) the adjoint variable $\psi(t) = a_0 + \lambda e^{-\rho t}/\rho$ in normal case, where $\lambda > 0$. Overtaking optimal control

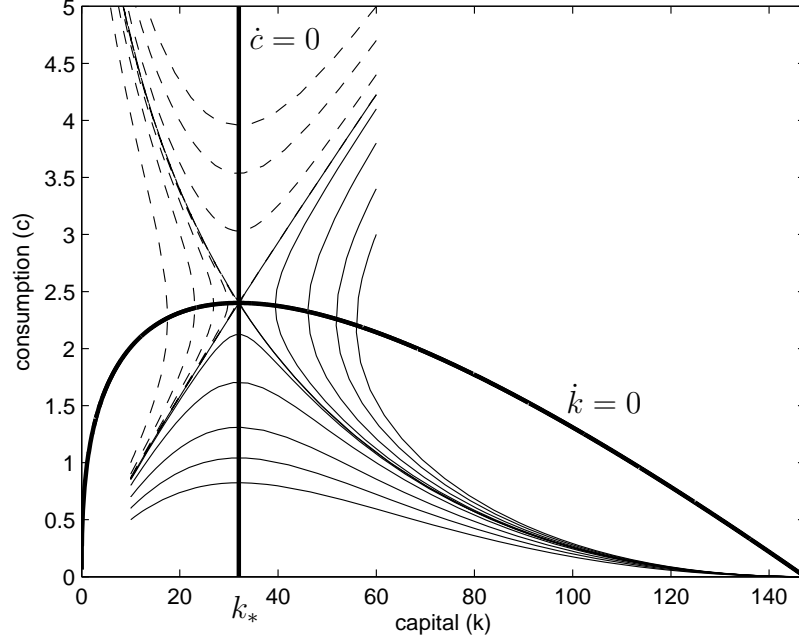


Figure 1: Phase diagram for $\alpha = 0.4$, $\delta = 0.05$, and $\theta = 0.5$. Bold lines denote stationary points, where $\dot{k} = 0$ and $\dot{c} = 0$. Solid lines are the trajectories satisfying optimality conditions in the form of Euler equation. Dashed lines are those trajectories which eventually violate nonnegativity of capital k .

$u \equiv 1$ provides maximum to the Hamiltonian $\mathcal{H}(x, u, t, \psi, \lambda) = \lambda e^{-\rho t} x + \psi u$ for all $t \geq 0$ if, and only if, $a_0 \geq 0$. Thus, $\hat{\psi}(t) = \lambda e^{-\rho t} / \rho$ resulting from (19) is also valid. Notice that in the abnormal case of $\lambda = 0$ the adjoint is strictly positive, $\psi \equiv a_0 > 0$, since $(\lambda, \psi_0) \neq 0$.

For $\rho = 0$ maximum principle holds only in the abnormal form, $\psi \equiv a_0 > 0$, otherwise ($\lambda > 0$) maximum condition (11) would be violated for any solutions $\psi(t) = a_0 - \lambda t$. In this case all conditions (1)–(4) are not satisfied.

All aforementioned adjoint solutions, including abnormal case, satisfy our general transversality condition (18) that reads as $\psi_0 \geq 0$, since $K \equiv 1$ and $u \leq 1$.

Example 5.3 ([4, Example 1.2], if $b = 0$). Let us maximize the following

integral

$$\max_u \int_0^\infty (x_2(t) + bu(t)) dt,$$

where $b > 0$, subject to the system describing a linear oscillator

$$\dot{x}_1(t) = x_2(t), \quad x_1(0) = 0, \quad (25)$$

$$\dot{x}_2(t) = u(t) - x_1(t), \quad x_2(0) = 0, \quad (26)$$

with bounded control $u(t) \in [-1, 1]$. We have the state-transition matrix

$$K(t, \tau) = \left(\exp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{t-\tau} = \begin{pmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{pmatrix}.$$

and the state trajectory

$$x(t) = \int_0^t K(t, s) \begin{pmatrix} 0 \\ u(s) \end{pmatrix} ds = \int_0^t u(s) \begin{pmatrix} \sin(t-s) \\ \cos(t-s) \end{pmatrix} ds.$$

Functional (7) reads as follows

$$J(u(\cdot), 0, 0, T) = x_1(T) + \int_0^T bu(t) dt = \int_0^T (\sin(T-t) + b) u(t) dt. \quad (27)$$

For $b \geq 1$ control $\hat{u} \equiv 1$ is overtaking optimal. For $b \in (0, 1)$ control $\hat{u} \equiv b$ is weakly overtaking optimal, see proof in the Appendix C. This control maximizes the Hamiltonian

$$\mathcal{H}(x, u, t, \psi, \lambda) = \lambda(x_2 + bu) + \psi_1 x_2 + \psi_2(u - x_1), \quad (28)$$

only in normal case ($\lambda = 1$) and only for solutions

$$\psi_1(t) = -r \cos(t + \phi) - 1, \quad \psi_2(t) = r \sin(t + \phi), \quad (29)$$

of the adjoint system

$$\dot{\psi}_1(t) = \psi_2(t), \quad (30)$$

$$\dot{\psi}_2(t) = -\psi_1(t) - \lambda, \quad (31)$$

where $|r| \leq b$ and ϕ is any phase shift. Indeed, in the abnormal case ($\lambda = 0$) control $\hat{u} \equiv 1$ would maximize Hamiltonian (28) only for $\psi_1 \equiv \psi_2 \equiv 0$, that would contradict $(\lambda, \psi_0) \neq 0$. Optimal trajectory reads as

$$\hat{x}_1(t) = 1 - \cos(t), \quad \hat{x}_2(t) = \sin(t). \quad (32)$$

Vector-function \hat{J}_x , defined in (15), reads as

$$\begin{aligned} \hat{J}_x(\tau, T) &= \int_{\tau}^T \begin{pmatrix} \cos(t - \tau) & -\sin(t - \tau) \\ \sin(t - \tau) & \cos(t - \tau) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt \\ &= \begin{pmatrix} \cos(T - \tau) - 1 \\ \sin(T - \tau) \end{pmatrix}. \end{aligned} \quad (33)$$

We denote $\Delta \hat{\mathcal{H}}(u, \tau, T) = \mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1)$. Since $\Delta \hat{\mathcal{H}}(u, \tau, T) = (u - 1)(\sin(T - \tau) + b)$, transversality condition (17) for WOO takes the form $(u - 1)(1 + b) \leq 0$ for all $u \in [-1, 1]$. Transversality condition (18) for OO reads as $(u - 1)(-1 + b) \leq 0$ for all $u \in [-1, 1]$. Let us check transversality conditions (1)–(4):

1. Adjoint vector $\psi(t)$, obtained in (29), does not converge to zero as $t \rightarrow \infty$.
2. Due to (32) the scalar product $\langle \psi(t), \hat{x}(t) \rangle \rightarrow 0$ as $t \rightarrow \infty$ only for $\psi_1(t) = -\cos(t) - 1$ and $\psi_2(t) = \sin(t)$ which belong to the correct adjoints in (29) only if $b \geq 1$.
3. Due to (32) and (29) Hamiltonian $\mathcal{H}(\hat{x}(t), \hat{u}(t), t, \psi(t), 1) = r \sin(\phi) + b$ can converge to zero only if $r \sin(\phi) = -b$. Since $|r| \leq b$ we have the only such adjoint solution: $\psi_1(t) = -b \sin(t) - 1$ and $\psi_2(t) = -b \cos(t)$.
4. Vector-function \hat{J}_x , obtained in (33), does not converge as $T \rightarrow \infty$.

A Proof of Proposition 4.1

Proof. **Needle variation** at time τ can be defined as

$$u_{\alpha}(t) := \begin{cases} \hat{u}(t), & t \notin (\tau - \alpha, \tau] \\ u & , \quad t \in (\tau - \alpha, \tau] \end{cases}, \quad (34)$$

where $u \in U$ is some constant. It is implied that $u = u(\tau)$.

The corresponding increment in the value of the functional can be written as follows:

$$\begin{aligned}\Delta J_T(\alpha) &:= J(u_\alpha(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T) \\ &= \int_{\tau-\alpha}^T (g(x_\alpha(t), \hat{u}(t), t) - g(\hat{x}(t), \hat{u}(t), t)) \, dt \\ &= J(\hat{u}(\cdot), x_\alpha(\tau), \tau, T) - J(\hat{u}(\cdot), \hat{x}(\tau), \tau, T) \\ &\quad + \int_{\tau-\alpha}^{\tau} (g(x_\alpha(t), u, t) - g(\hat{x}(t), \hat{u}(t), t)) \, dt,\end{aligned}$$

where x_α is the trajectory corresponding to control u_α . Then, due to Assumption 1, we have that for all $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that for all $T \geq \tau$ the following inequality holds

$$\begin{aligned}\frac{\Delta J_T(\alpha)}{\alpha} &\geq -\varepsilon + \left\langle \hat{J}_x(\tau, T), \zeta \right\rangle \\ &\quad + \frac{1}{\alpha} \int_{\tau-\alpha}^{\tau} (g(x_\alpha(t), u, t) - g(\hat{x}(t), \hat{u}(t), t)) \, dt.\end{aligned}\tag{35}$$

Approximate solution: We take the vector ζ in (35) as

$$\zeta_\alpha(\tau) = \frac{x_\alpha(\tau) - \hat{x}(\tau)}{\alpha}.$$

Due to the differentiability of function f with respect to x we have from differential equation (6) the following limit

$$\lim_{\alpha \rightarrow 0} \zeta_\alpha(\tau) = y(\tau),\tag{36}$$

where y is the solution (14) of the linearized system (13), $y(t) = K(t, \tau) y(\tau)$, with the following initial condition at $t = \tau$:

$$y(\tau) = f(\hat{x}(\tau), u, \tau) - f(\hat{x}(\tau), \hat{u}(\tau), \tau).\tag{37}$$

Taking into account expression (36), we have the following approximation of the last term in (35):

$$\begin{aligned}&\frac{1}{\alpha} \int_{\tau-\alpha}^{\tau} (g(x_\alpha(t), u, t) - g(\hat{x}(t), \hat{u}(t), t)) \, dt \\ &= g(\hat{x}(\tau), u, \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) + O(\alpha),\end{aligned}$$

where $\lim_{\alpha \rightarrow 0} O(\alpha) = 0$. Hence, inequality (35) takes the form

$$\frac{\Delta J_T(\alpha)}{\alpha} \geq -\varepsilon + \left\langle \hat{J}_x(\tau, T), \zeta_\alpha(\tau) \right\rangle + g(\hat{x}(\tau), u, \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) + O(\alpha).$$

Limit (36) implies that for all $\varepsilon > 0$ there exists $\tilde{\alpha}(\varepsilon) > 0$ such that

$$|y(\tau) - \zeta_\alpha(\tau)| \leq \frac{\varepsilon}{M(\tau)}, \quad \text{for all } \alpha \in (0, \tilde{\alpha}(\varepsilon)]. \quad (38)$$

We have from (16) and (38) that

$$\left\langle \hat{J}_x(\tau, T), y(\tau) - \zeta_\alpha(\tau) \right\rangle \leq \left| \hat{J}_x(\tau, T) \right| |y(\tau) - \zeta_\alpha(\tau)| \leq \varepsilon,$$

and

$$\left\langle \hat{J}_x(\tau, T), y(\tau) \right\rangle \leq \left\langle \hat{J}_x(\tau, T), \zeta_\alpha(\tau) \right\rangle + \varepsilon.$$

Hence, we have inequality (35) in the form

$$\frac{\Delta J_T(\alpha)}{\alpha} \geq -2\varepsilon + \left\langle \hat{J}_x(\tau, T), y(\tau) \right\rangle + g(\hat{x}(\tau), u, \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) + O(\alpha).$$

Definition 3.2 of WOO means, that for all $\varepsilon_2 > 0$ and $T_2 > t_0$ there exists $T'(\varepsilon_2) \geq T_2$ such that holds $\Delta J_{T'}(\alpha) \leq \alpha \varepsilon_2$. Let us take $T_2 \geq T_1$ and $\varepsilon_2 = \alpha \varepsilon$. Then inequality $\Delta J_{T'}(\alpha) \leq \alpha \varepsilon$ results in

$$3\varepsilon \geq + \left\langle \hat{J}_x(\tau, T'), y(\tau) \right\rangle + g(\hat{x}(\tau), u, \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) + O(\alpha). \quad (39)$$

Suppose that (17) is violated, i.e. there exist $\varepsilon > 0$ and $T \geq t_0$ such that for all $T' \geq T$

$$\left\langle \hat{J}_x(\tau, T'), y(\tau) \right\rangle + g(\hat{x}(\tau), u, \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) \geq 4\varepsilon,$$

Then we have contradiction with (39) taking α small enough, i.e (17) should hold.

Similar calculations can be done for the transversality condition (18) for OO. \square

B Proof of Proposition 4.2

First, we prove the following Lemma.

Lemma B.1. *Adjoint equation (12) can be written with the use of (15) as*

$$\psi(\tau) = K^*(T, \tau) \psi(T) + \lambda \hat{J}_x(\tau, T), \quad \psi(t_0) = \psi_0. \quad (40)$$

Proof. Since vector $y(\tau)$ in (14) is arbitrary, from (13) one can find the matrix derivative

$$\frac{\partial K}{\partial t}(t, \tau) = \left(\frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t) \right) K(t, \tau).$$

Taking the Hermitian transpose we have

$$\frac{\partial K^*}{\partial t}(t, \tau) = K^*(t, \tau) \left(\frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t) \right)^*.$$

Hence, if we multiply the adjoint equation (12) by matrix $K^*(t, \tau)$ as

$$-K^*(t, \tau) \dot{\psi}(t) = K^*(t, \tau) \left(\frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t) \right)^* \psi(t) + \lambda K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t),$$

then we have

$$-\frac{\partial}{\partial t} (K^*(t, \tau) \psi(t)) = \lambda K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t).$$

Integration of the latter equation from τ till T yields (40). \square

B.1 Main proof

Proof. 1) It follows from (21) and (40) with $\lambda = 1$ that

$$\begin{aligned} \psi(\tau) - \lim_{T \rightarrow +\infty} \hat{J}_x(\tau, T) &= \lim_{T \rightarrow +\infty} K^*(T, \tau) \psi(T) \\ &= K^*(t_0, \tau) \lim_{T \rightarrow +\infty} (K^*(T, t_0) \psi(T)) \\ &= K^*(t_0, \tau) a_0, \end{aligned}$$

where we use the expression $K^*(T, \tau) = (K(T, t_0) K(t_0, \tau))^* = K^*(t_0, \tau) K^*(T, t_0)$.

Taking into account (19) we have $\psi(\tau) - \hat{\psi}(\tau) = K^*(t_0, \tau) a_0$.

2) It follows from (40) with $\lambda = 0$, that $\psi(\tau) = K^*(t_0, \tau) a_0$. \square

C Proof of optimality in example 5.2

We show that for $b \in (0, 1)$ control $u \equiv 1$ is weak overtaking optimal among all controls in $[0, 1]$, see Definition 3.2, where

$$J(u(\cdot), 0, 0, T) - J(1, 0, 0, T) = \int_0^T (\sin(T - t) + b) (u(t) - 1) dt. \quad (41)$$

It suffice to prove that for the function

$$\Delta x_1(T) := \int_0^T \sin(T - t) (u(t) - 1) dt \quad (42)$$

for all $T' \geq 0$ there exists $T \geq T'$ such that $\Delta x_1(T) \leq 0$. Assume the opposite, i.e. there exists $T' \geq 0$ such that for all $T \geq T'$ holds $\Delta x_1(T) > 0$. Take the integer number n such that $2\pi(n - 1) > T'$, so that

$$\Delta x_1(2\pi(n - 1)) = \int_0^{2\pi(n-1)} \sin(t) (u(t) - 1) dt > 0.$$

The function Δx_1 of $T = 2n\pi$ can be written as

$$\begin{aligned} \Delta x_1(2n\pi) &= - \int_0^{2n\pi} \sin(t) (u(t) - 1) dt \\ &= - \Delta x_1(2\pi(n - 1)) - \int_{(2n-1)\pi}^{2n\pi} \sin(t) (u(t) - 1) dt. \end{aligned} \quad (43)$$

The last integral is not negative

$$\int_{(2n-1)\pi}^{2n\pi} \sin(t) (u(t) - 1) dt \geq 0,$$

since $u(t) \leq 1$ and $\sin(t) \leq 0$ for all $t \in [(2n - 1)\pi, 2n\pi]$. Thus, recalling that $\Delta x_1(2\pi(n - 1)) > 0$, we have inequality $\Delta x_1(2n\pi) < 0$, which contradicts the assumption. \square

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